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COMPLETE CONFLICT CONTROLLABILITY OF QUASILINEAR PROCESSES*

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The problem of guaranteed arrival of the trajectory of a quasilinear conflict-controlled process (CCP) to a terminal set from any initial position is investigated. The solving function method /1-3/ is used to derive the sufficient conditions of solvability of the problem. The results are illustrated with some examples.

CCP controllability has been very little studied; only a small number of references can be cited, in particular /4, 5/.

1. A CCP is defined by the quasilinear differential equation

$$z^{\cdot} = Az + \varphi(u, v),\ z \in \mathbb{R}^n,\ u \in U,\ v \in V \tag{1.1}$$

where U and V are non-empty compact sets in finite-dimensional spaces and $\varphi(u, v)$ is a function jointly continuous in all its variables. The terminal set has the form $M^* = M + M^\circ$, where M° is a linear subspace of R^n and M is a compactum in the orthogonal complement L of M°.

We say that the CCP (1.1) is completely controllable if for any initial position $z^\circ \in R^n$ there exists a time $T(z^\circ) < \infty$ and a measurable function

$$u(t) = u(z^\circ, v_t(\cdot)) \in U\ (v_t(\cdot) = \{v(s) \in V\colon\ s \in [0, t]\})$$

such that the solution of Eq.(1.1) reaches the set M^* not later than the time $T(z^\circ)$ for any measurable function $v(t) \in V,\ t \in [0, T(z^\circ)]$.

Denote by π the orthogonal projection operator from R^n to L.

Condition 1°. A number $a > 0$ exists such that $\|\pi e^{At}\| \leqslant a$ for all $t \geqslant 0$.

We define the multivalued mappings

$$\Phi(t, \tau, v) = \pi e^{A(t-\tau)}\varphi(U, v) - M(t, \tau),\ \Phi(t, \tau) = \bigcap_{v \in V} \Phi(t, \tau, v) \tag{1.2}$$

where $M(\cdot)$ is some multivalued mapping $R \times R \to 2^L$.

Condition 2°. A compact-valued mapping $M(t, \tau)$ exists, measurable with respect to τ, such that

1) $0 \in \Phi(t, \tau),\ \forall t \geqslant \tau \geqslant 0$

2) $\int_0^t M(t, \tau)\, d\tau \subset M,\ \ \forall t \geqslant 0$

3) A number $\mu > 0$ exists such that

$$\int_0^t \|M(t, \tau)\|\, d\tau \leqslant \mu,\ \ \forall t \geqslant 0;\ \ \|M(t, \tau)\| = \max_{m \in M(t, \tau)} \|m\|$$

Remark 1. If M is a convex compactum, then an appropriate compact-valued mapping $M(t, \tau)$ can be sought in the form $\omega(t, \tau)\cdot M$ /2, 3/, where $\omega(t, \tau)$ is a non-negative τ-measurable function such that

$$\int_0^t \omega(t, \tau)\, d\tau = 1$$

This multivalued mapping automatically satisfies parts 2 and 3 of Condition 2°.

If Conditon 2° is satisfied, then the function

$$\rho(t, \tau, v, z) = \max\{\rho \geqslant 0\colon \rho\cdot z \in \Phi(t, \tau, v)\},\ t \geqslant \tau \geqslant 0,\ z \in L,\ z \neq 0$$

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is defined (the inverse of the Minkowski functional $\Phi(t, \tau, v)$. Let

$$\alpha(t, \tau, z) = \inf_{v \in V} \rho(t, \tau, v, z) \tag{1.3}$$

We know /6/ that for any matrix A there exists a decomposition of the space R^n into a direct sum of linear subspaces invariant with respect to A which correspond to the eigenvalues with positive, zero, and negative real parts respectively: $R^n = R_+ + R_0 + R_-$.

In the subspace R_0, we can isolate the subspace R_1 spanned by the eigenvectors that correspond to eigenvalues with zero real part. Note that the operator e^{At} is bounded uniformly in $t \geqslant 0$ on the invariant subspace $R_1 + R_-$, and Condition 1° implies the inclusion $R_+ \subset M^\circ$.

Put $S = \{z \in L: \|z\| = 1\}$, $D = \{z \in L: \|z\| \leqslant 1\}$, $S_0 = S \cap \pi R_0$.

Condition 3°. Numbers $\varepsilon > 0$, $\theta > 0$ exist such that the set

$$T(\varepsilon, \theta) = \left\{t \geqslant 0: \inf_{s \in S} \int_0^\theta \alpha(t, \tau, s)\, d\tau \geqslant \varepsilon\right\}$$

is non-empty and unbounded, and

$$\sup_{t \in T(\varepsilon, \theta)} \inf_{s \in S_0} \int_0^t \alpha(t, \tau, s)\, d\tau = \infty \tag{1.4}$$

If $S_0 = \varnothing$ (i.e., $R_0 \subset M^\circ$), then we formally assume that equality (1.4) holds.

Condition 4°.

$$\limsup_{t \to \infty} \inf_{s \in S} \int_0^t \alpha(t, \tau, s)\, d\tau > 0$$

Theorem 1. If the CCP (1.1) satisfies Conditions 1°, 2°, and 3°, then it is completely controllable.

Proof. For any initial position $z^\circ \in R^n$, we have the unique decomposition $z^\circ = z_+^\circ + z_0^\circ + z_-^\circ$, where $z_+^\circ \in R_+$, $z_0^\circ \in R_0$, $z_-^\circ \in R_-$.

By the definition of z_-° there exists a time $T_1 = T_1(z_-^\circ) < \infty$ such that $T_1 \geqslant \theta$ and $\|\pi e^{At} z_-^\circ\| \leqslant \varepsilon$ for all $t \geqslant T_1$. By Condition 3°, there exists a time $T < \infty$ such that $T \geqslant T_1$ and

$$\inf_{s \in S} \int_0^\theta \alpha(T, \tau, s)\, d\tau \geqslant \varepsilon \tag{1.5}$$

$$\inf_{s \in S_0} \int_0^T \alpha(T, \tau, s)\, d\tau \geqslant a\|z_0^\circ\| + a\|\varphi(U, V)\| T_1 + \mu \tag{1.6}$$

$$\|\varphi(U, V)\| = \max_{u \in U,\, v \in V} \|\varphi(u, v)\|$$

where μ is the number from Condition 2°. This $T = T(z^\circ)$ is the guaranteed arrival time of the trajectory of the CCP (1.1) in the terminal set M^* from the initial position z°. Let us prove this statement.

Put $\Pi_\xi = \pi e^{AT} z_\xi^\circ$, $\xi = +, 0, -$. From the definition of $T_1(z_-^\circ)$ and condition (1.5) we have

$$\inf_{s \in S} \int_0^{T_1} \alpha(T, \tau, s)\, d\tau \geqslant \varepsilon \geqslant \|\Pi_-\| \tag{1.7}$$

For any $s \in S$ and $t \geqslant \tau \geqslant 0$ we have the bounds

$$\alpha(t, \tau, s) \leqslant \max_{u \in U,\, v \in V,\, m \in M(t, \tau)} \|\pi e^{A(t-\tau)} \varphi(u, v) - m\| \leqslant$$
$$\|\pi e^{A(t-\tau)}\| \|\varphi(U, V)\| + \|M(t, \tau)\|$$

Therefore for all $s \in S$ and $t \geqslant T_1$,

$$\int_0^{T_1} \alpha(t, \tau, s)\, d\tau \leqslant a\|\varphi(U, V)\| T_1 + \mu < \infty$$

From this bound and condition (1.6) we obtain

$$\inf_{s \in S_0} \int_{T_1}^{T} \alpha(T, \tau, s)\, d\tau \geqslant a \| z_0^\circ \| \geqslant \| \Pi_0 \| \tag{1.8}$$

Using the obvious identity $\lambda\rho(T, \tau, \lambda z, v) = \rho(T, \tau, z, v)$ (for all $\lambda > 0$, $T \geqslant \tau \geqslant 0$, $z \in L$, $z \neq 0$, $v \in V$) and assuming that $\Pi_0 \neq 0$ and $\Pi_- \neq 0$, we obtain from (1.7) and (1.8)

$$\int_0^{T_1} \alpha(T, \tau, -\Pi_-)\, d\tau = \frac{1}{\|\Pi_-\|} \int_0^{T_1} \alpha\left(T, \tau, -\frac{\Pi_-}{\|\Pi_-\|}\right) d\tau \geqslant 1 \tag{1.9}$$

$$\int_{T_1}^{T} \alpha(T, \tau, -\Pi_0)\, d\tau \geqslant 1 \tag{1.10}$$

For an arbitrary measurable function $v(t) \in V$ choose the control $u(t) \in U$ and the function $m(t) \in M(T, t)$ $(t \in [0, T])$ from the system of equations

$$\pi e^{A(T-t)} \varphi(u(t), v(t)) - m(t) = \begin{cases} \rho(T, t, -\Pi_-, v(t))(-\Pi_-), & t \in [0, \theta_1] \\ \rho(T, t, -\Pi_0, v(t))(-\Pi_0), & t \in [T_1, \theta_2] \\ 0, & t \in (\theta_1, T_1) \cup (\theta_2, T] \end{cases} \tag{1.11}$$

where the times θ_1 and θ_2, which exist by inequalities (1.9) and (1.10), are determined from the conditions

$$\int_0^{\theta_1} \rho(T, t, -\Pi_-, v(t))\, dt = \int_{T_1}^{\theta_2} \rho(T, t, -\Pi_0, v(t))\, dt = 1$$

By the Filippov-Kasten theorem /7/, the system of Eqs.(1.11) is solvable in the class of measurable functions $u(t) \in U$, $m(t) \in M(T, t)$, $t \in [0, T]$.

Note that from Condition 1° we obtain $\Pi_+ = 0$.

Choosing the measurable control $u(t) \in U$ $(t \in [0, T])$ from the system (1.11), we obtain

$$\begin{gathered} \pi z(T) = \Pi_+ + \Pi_0 + \Pi_- + \int_0^T (\pi e^{A(T-t)} \varphi(u(t), v(t)) - m(t))\, dt + \int_0^T m(t)\, dt = \\ \Pi_+ + \int_0^T m(t)\, dt + \Pi_0 + \int_{T_1}^{\theta_2} \rho(T, t, -\Pi_0, v(t))\, dt \cdot (-\Pi_0) + \\ \Pi_- + \int_0^{\theta_1} \rho(T, t, -\Pi_-, v(t))\, dt \cdot (-\Pi_-) \in \int_0^T M(T, t)\, dt + 0 \subset M \end{gathered} \tag{1.12}$$

To complete the proof of the theorem, note that if $\Pi_- = 0$, then the measurable control $u(t) \in U$ $(t \in [0, T])$ should be chosen from the equation

$$\pi e^{A(T-t)} \varphi(u(t), v(t)) - m(t) = \begin{cases} \rho(T, t, -\Pi_0, v(t))(-\Pi_0), & t \in [T_1, \theta_2] \\ 0, & t \in [0, T_1) \cup (\theta_2, T] \end{cases}$$

and if $\Pi_0 = 0$, then it should be chosen from the equation

$$\pi e^{A(T-t)} \varphi(u(t), v(t)) - m(t) = \begin{cases} \rho(T, t, -\Pi_-, v(t))(-\Pi_-), & t \in [0, \theta_1] \\ 0, & t \in (\theta_1, T] \end{cases}$$

The theorem is proved for this case in the same way as (1.12).

Corollary 1. If Conditions 2° and 3° are satisfied, the trajectory of the CCP (1.1) can be taken to the terminal set from any initial position $z^\circ \in R_1 \cup R_-$.

Proof. If $z^\circ \in R_1$, then there exists a number $d > 0$ such that $\| \pi e^{At} z^\circ \| \leqslant d$ for all $t \geqslant 0$.

By Condition 3°, there exists a time T such that

$$\inf_{s\in S_0}\int_0^T \alpha(T,\tau,s)\,d\tau \geqslant d$$

If $z^0 \in R_-$, then there exists a time $T \geqslant \theta$ such that $\|\Pi\| \leqslant \varepsilon$ and

$$\inf_{s\in S}\int_0^T \alpha(T,\tau,s)\,d\tau \geqslant \varepsilon$$

Thus, in either case there is a time $T = T(z^0)$ such that

$$\inf_{v(\cdot)\in V}\int_0^T \rho(T,\tau,-\Pi,v(\tau))\,d\tau \geqslant 1$$

For an arbitrary measurable function $v(t) \in V\ (t\in[0,T])$, the measurable control $u(t)\in U$ which takes the trajectory of the CCP (1.1) from the initial state z^0 to the terminal set M^* at the time T is chosen from the system of equations

$$\pi e^{A(T-t)}\varphi(u(t),v(t)) - m(t) = \begin{cases}\rho(T,t,-\Pi,v(t))(-\Pi), & t\in[0,\theta_1]\\ 0, & t\in(\theta_1,T]\end{cases}$$

where the time $\theta_1 \in [0,T]$ is defined by the equality

$$\int_0^{\theta_1} \rho(T,\tau,-\Pi,v(\tau))\,d\tau = 1$$

and $m(t)$ is a measurable selector of $M(T,t), t\in[0,T]$.

The rest of the proof is similar to the proof of the theorem.

Corollary 2. Assume that the real parts of all eigenvalues of the restriction of the operator A to the subspace L are negative (i.e., $R_0 + R_+ \subset M^0$) and that Condition 2° and 4° are satisfied for the CCP (1.1). Then the process (1.1) is completely controllable.

Proof. By Condition 4°.

$$\eta = \limsup_{t\to\infty}\inf_{s\in S}\int_0^t \alpha(t,\tau,s)\,d\tau > 0$$

Thus, values of T as large as desired exist such that

$$\inf_{s\in S}\int_0^T \alpha(T,\tau,s)\,d\tau \geqslant \frac{\eta}{2} \tag{1.13}$$

At the same time, by the condition of the corollary, for the initial position $z^0 \in R^n$ there exists a number $T_1 = T_1(z^0) < \infty$ such that $\|\pi e^{At}z^0\| \leqslant \eta/2$ for all $t \geqslant T_1$.

Then the time $T = T(z^0) \geqslant T_1$ satisfying condition (1.13) is the required moment of guaranteed arrival of the trajectory of the CCP (1.1) from the initial position z^0 in the terminal set. This assertion is proved in the same way as Theorem 1.

Let

$$g_1(t,\tau) = \inf_{s\in S}\ \alpha(t,\tau,s),\quad g_2(t,\tau) = \inf_{s\in S_0}\ \alpha(t,\tau,s) \tag{1.14}$$

Condition 5°. Numbers $\varepsilon > 0,\ \theta > 0$ exists such that the set

$$T(\varepsilon,\theta) = \left\{t \geqslant 0: \int_0^\theta g_1(t,\tau)\,d\tau \geqslant \varepsilon\right\}$$

is non-empty and unbounded and

$$\sup_{t\in T(\varepsilon,\theta)} \int_0^t g_2(t,\tau)\,d\tau = \infty$$

Condition 5° obviously implies Condition 3°, and we thus have the following proposition.

Corollary 3. If the CCP (1.1) satisfies Conditions 1°, 2°, and 5°, then it is completely controllable.

2. Consider a different approach to controllability analysis of the process (1.1). We define the multivalued mappings

$$W(t,v) = \pi e^{At}\varphi(U,v), \quad W(t) = \bigcap_{v\in V} W(t,v)$$

Condition 6°. $0 \in W(t)$ for all $t \geqslant 0$.

Let

$$\sigma(t,z,v) = \max\{\sigma \geqslant 0\colon \sigma\cdot z \in W(t,v)\} \tag{2.1}$$

$$\beta(t,z) = \inf_{v\in V} \sigma(t,z,v), \quad z\in L, \quad z\neq 0, \quad t\geqslant 0 \tag{2.2}$$

Condition 7°.

$$1)\ \sup_{t\geqslant 0}\inf_{s\in S}\int_0^t \beta(\tau,s)\,d\tau > 0, \qquad 2)\ \sup_{t\geqslant 0}\inf_{s\in S_0}\int_0^t \beta(\tau,s)\,d\tau = \infty.$$

Note that (as in Condition 3°) if $S_0 = \varnothing$, then we formally assume that part 2 of Condition 7° holds.

Theorem 2. If $M^* = m + M^\circ$ is an affine manifold and Conditions 1°, 6°, and 7° are satisfied, then the CCP (1.1) is completely controllable for an arbitrary vector $m \in R_+ + R_0$.

Proof. As in the proof of Theorem 1 note that for any initial position $z^\circ \in R^n$ there is a unique decomposition $z^\circ = z_+^\circ + z_0^\circ + z_-^\circ$, $z_+^\circ \in R_+$, $z_0^\circ \in R_0$, $z_-^\circ \in R_-$.

By part 1 of Condition 7°, there exist $\varepsilon > 0$ and $\theta > 0$ such that

$$\inf_{s\in S}\int_0^\theta \beta(\tau,s)\,d\tau \geqslant \varepsilon$$

Using these ε,θ, and z_-° we determine the time T_1 such that $T_1\geqslant\theta$ and $\|\Pi_-\|\leqslant\varepsilon$ for all $t\geqslant T_1$. Then from part 2 of Condition 7° it follows that the time $T = T(z^\circ)$ exists such that

$$\inf_{s\in S_0}\int_0^T \beta(\tau,s)\,d\tau \geqslant a\cdot\|\varphi(U,V)\|\cdot T_1 + \|\pi m\| + a\|z_0^\circ\| \geqslant \inf_{s\in S_0}\int_0^{T_1}\beta(\tau,s)\,d\tau + \|\pi m - \Pi_0\|$$

where in the second inequality we have used the bound

$$\beta(\tau,s) \leqslant \max_{u\in U,\, v\in V}\|\pi e^{A\tau}\varphi(u,v)\| \leqslant a\|\varphi(U,V)\|, \quad s\in S,\ \tau\geqslant 0$$

The time $T(z^\circ)$ is the guaranteed time of arrival of the trajectory of the process (1.1) in the terminal set $\{m + M^\circ\}$ from the initial position z°. Let us prove this statement.

Assuming that $\Pi_-\neq 0$ and $\pi m - \Pi_0 \neq 0$, we obtain from the preceding argument

$$\int_0^{T-T_1} \beta(T-\tau,\ \pi m - \Pi_0)\,d\tau = \int_{T_1}^T \beta(\tau,\pi m - \Pi_0)\,d\tau \geqslant 1 \tag{2.3}$$

$$\int_{T-T_1}^T \beta(T-\tau,\ -\Pi_-)\,d\tau = \int_0^{T_1}\beta(\tau,-\Pi_-)\,d\tau \geqslant 1 \tag{2.4}$$

For an arbitrary measurable function $v(t)\in V$, we choose the measurable control $u(t)\in U$ $(t\in[0,T])$ from the system of equations

$$\pi e^{A(T-\tau)}\varphi(u(\tau),v(\tau)) = \begin{cases} \sigma(T-\tau,\ \pi m - \Pi_0,\ v(\tau))\,(\pi m - \Pi_0), & \tau\in[0,\theta_1] \\ \sigma(T-\tau,\ -\Pi_-,\ v(\tau))\,(-\Pi_-), & \tau\in[T-T_1,\theta_2] \\ 0, & \tau\in(\theta_1,\ T-T_1)\cup(\theta_2,T] \end{cases}$$

where the times θ_1 and θ_2, which exist by inequalities (2.3) and (2.4), are determined from the conditions

$$\int_0^{\theta_1} \sigma(T-\tau,\ \pi m - \Pi_0,\ v(\tau))\, d\tau = \int_{T-T_1}^{\theta_2} \sigma(T-\tau,\ -\Pi_-, v(\tau))\, d\tau = 1$$

Then we obtain

$$\pi z(T) = \Pi_+ + \Pi_0 + \Pi_- + \int_0^T \pi e^{A(T-\tau)} \varphi(u(\tau), v(\tau)) d\tau = \Pi_+ + \Pi_0 +$$
$$\int_0^{\theta_1} \sigma(T-\tau, \pi m - \Pi_0, v(\tau))\, d\tau \cdot (\pi m - \Pi_0) + \Pi_- +$$
$$\int_{T-T_1}^{\theta_2} \sigma(T-\tau, -\Pi_-, v(\tau))\, d\tau \cdot (-\Pi_-) = \pi m$$

To conclude the proof, note that the cases $\Pi_- = 0$ and $\Pi_0 = \pi m$ are analysed as in Theorem 1.

Let

$$h_1(t) = \inf_{s \in S} \beta(t, s), \quad h_2(t) = \inf_{s \in S_0} \beta(t, s). \tag{2.5}$$

Condition 8°.

1) $\sup_{t \geqslant 0} \int_0^t h_1(\tau)\, d\tau > 0,$ 2) $\sup_{t \geqslant 0} \int_0^t h_2(\tau)\, d\tau = \infty.$

Corollary 4. If the CCP (1.1) satisfies Conditions 1°, 6°, and 8°, then it is completely controllable.

Corollary 5. Assume that the real parts of all eigenvalues of the restriction of the operator A to the subspace L are negative, and also that Condition 6° holds and $0 \in \operatorname{int} M$ (int is the interior of a set). Then the CCP (1.1) is completely controllable.

Proof. Since $0 \in \operatorname{int} M$, then $\delta D \subset M$ for some $\delta > 0$. Then the multivalued mapping $M(t, \tau) = \delta / t D\ (t > 0)$ satisfies Condition 2° and for the function (1.3) corresponding to this mapping we have the inequality $\alpha(t, \tau, s) \geqslant \delta / t\ (s \in S)$, so that Condition 4° is satisfied. Thus, all the assumptions of Corollary 2 are satisfied, and the process (1.1) is completely controllable.

Note that a proposition similar to Corollary 5 is given in /5/ (for a somewhat less-general case).

3. The conditions of complete controllability of the CCP (1.1) given in the previous sections ensure the arrival of the trajectory in the terminal set at a fixed time. This result strongly relies on analogues of Pontryagin's condition /8/ (Conditions 2° and 6°). Let us now consider the problem of the controllability of the process (1.1) with a free arrival time of the trajectory in the terminal set, abandoning Conditions 2° and 6°.

Note that for the function $\beta(t, s)$ (2.2) to be well-defined, we do not necessarily need Condition 6°. It is sufficient that the function $\sigma(t, s, v)$ (2.1) $(t \geqslant 0,\ s \in S,\ v \in V)$ is defined. The function $\sigma(\cdot)$ in turn may be well-defined although Condition 6° does not hold.

Theorem 3. Assume that for the CCP (1.1) Condition 1° is satisfied, the function $\beta(t, s)$ $(t \geqslant 0,\ s \in S)$ is defined, Condition 7° is satisfied, and also $\pi A = A\pi$ and $0 \in M$. Then the CCP (1.1) is completely controllable.

Proof. By Condition 7° there exist $\varepsilon > 0$ and $\theta > 0$ such that

$$\inf_{s \in S} \int_0^\theta \beta(\tau, s)\, d\tau \geqslant \varepsilon$$

Using these ε, θ, and z_-° we determine a time T_1 such that $T_1 \geqslant \theta$ and $\|\Pi_-\| \leqslant \varepsilon$ for all $t \geqslant T_1$. There exists a time $T = T(z^\circ)$ such that

$$\inf_{s \in S_0} \int_0^T \beta(\tau, s)\, d\tau \geqslant a \|\varphi(U, V)\| T_1 + a \|z_0^\circ\|$$

Assuming that $\Pi_- \neq 0$ and $\Pi_0 \neq 0$, we choose the control $u(t) \in U$ ($t \in [0, T]$) for an arbitrary measurable function $v(t) \in V$ from the system of equations

$$\pi e^{A(T-t)} \varphi(u(t), v(t)) = \begin{cases} \sigma(T-t, -\Pi_0, v(t))(-\Pi_0), & t \in [0, \theta_1] \\ \sigma(T-t, -\Pi_-, v(t))(-\Pi_-), & t \in (\theta_1, T] \end{cases}$$

where the time θ_1 is defined by the condition

$$\int_0^{\theta_1} \sigma(T-\tau, -\Pi_0, v(\tau))\, d\tau = 1$$

Define the time $\theta_2 \in [\theta_1, T]$ by the condition

$$\int_{\theta_1}^{\theta_2} \sigma(T-\tau, -\Pi_-, v(\tau))\, d\tau = 1$$

For this time we have

$$\pi z(\theta_2) = \pi e^{A\theta_2}(z_+^\circ + z_0^\circ + z_-^\circ) + \int_0^{\theta_2} \pi e^{A(\theta_2-\tau)} \varphi(u(\tau), v(\tau))\, d\tau =$$

$$\pi e^{A\theta_2} z_+^\circ + e^{A(\theta_2 - T)} \Big\{ \Pi_0 + \int_0^{\theta_1} \pi e^{A(T-\tau)} \varphi(u(\tau), v(\tau))\, d\tau +$$

$$\Pi_- + \int_{\theta_1}^{\theta_2} \pi e^{A(T-\tau)} \varphi(u(\tau), v(\tau))\, d\tau \Big\} = 0 \in M$$

which it was required to prove. The cases $\Pi_- = 0$ and $\Pi_0 = 0$ are proved similarly.

Corollary 6. Assume that the CCP (1.1) satisfies the following conditions:

1) $A = \lambda E$, where E is the identity transformation in R^n, $\lambda \leqslant 0$;
2) $0 \in M$;
3) there exists a number $\varepsilon > 0$ such that

$$\max\{\sigma \geqslant 0\colon \sigma \cdot s \in \pi\varphi(U, v)\} \geqslant \varepsilon, \quad \forall v \in V, \quad \forall s \in S.$$

Then the CCP (1.1) is completely controllable.

4. Let us apply the results of the previous sections to study the linear pursuit problem

$$x^\cdot = Bx + u, \quad y^\cdot = Cy + v, \quad x \in R^n, \quad y \in R^n, \ u \in U, \quad v \in V$$

The terminal set is $M^* = \{(x, y) \in R^n \times R^n\colon\ x = y\}$, i.e., capture occurs when the phase coordinates are equal.

Let us reduce the problem to the form (1.1). Let $z_1 = x - y$, $z_2 = y$, $z = (z_1, z_2) \in R^{2n}$. Then

$$M^* = M^\circ = \{(z_1, z_2)\colon z_1 = 0\}, \quad \pi(z_1, z_2) = z_1$$

$$A = \begin{Vmatrix} B, & B - C \\ 0, & C \end{Vmatrix}, \quad e^{At} = \begin{Vmatrix} e^{Bt}, & e^{Bt} - e^{Ct} \\ 0, & e^{Ct} \end{Vmatrix},$$

$$e^{At} \begin{Vmatrix} u - v \\ v \end{Vmatrix} = \begin{Vmatrix} e^{Bt}u - e^{Ct}v \\ e^{Ct}v \end{Vmatrix}$$

$W(t) = e^{Bt}U \overset{*}{-} e^{Ct}V$ ($\overset{*}{-}$ denotes the geometrical difference of sets /8/), $\pi e^{At}z = e^{Bt}x - e^{Ct}y$.

Condition 9°. There exists a continuous function $\gamma(t)$, $0 \leqslant \gamma(t) \leqslant 1$, $t \geqslant 0$, such that $e^{Ct}V \subset \gamma(t)\, e^{Bt}U$.

Corollary 7. Assume that Condition 9° holds and also

1) there exist $b > 0$, $c > 0$ such that $\| e^{Bt} \| \leqslant b$, $\| e^{Ct} \| \leqslant c$ for all $t \geqslant 0$;
2) U is a convex compactum and $0 \in \operatorname{int} U$;
3) $\limsup\limits_{t\to\infty} \gamma(t) < 1$.

Then pursuit from the initial positions x° and y° may be terminated in a finite time.

Proof. By the convexity of U and Condition 9°, we have

$$W(t) = e^{Bt}U \overset{*}{-} e^{Ct}V \supset (1 - \gamma(t))\, e^{Bt}U, \quad t \geqslant 0$$

Condition 1 of the corollary ensures that Condition 1° is satisfied and also indicates that the operator B does not have eigenvalues with a positive real part and all eigenvalues with zero real part are simple, i.e., $R^n = R_1(B) + R_-(B)$.

By Condition 2 of the corollary, the compactum U includes a sphere εD of radius $\varepsilon > 0$. Then by the definition of the functions $h_1(\cdot)$ and $h_2(\cdot)$ (2.5), we have

$$h_1(t) \geqslant (1-\gamma(t))\,\varepsilon \inf_{\|s\|=1} \| e^{Bt}s \|, \quad h_2(t) \geqslant (1-\gamma(t))\,\varepsilon \inf_{s\in R_1(B),\, \|s\|=1} \| e^{Bt}s \|$$

For some $d > 0$ we have

$$\inf_{s\in R_1(B),\, \|s\|=1} \| e^{Bt}s \| = (\sup_{s\in R_1(B),\, \|s\|=1} \| e^{-Bt}s \|)^{-1} \geqslant d$$

and using Condition 3 of the corollary, we obtain

$$h_1(t) \geqslant \frac{(1-\gamma(t))\,\varepsilon}{\| e^{-Bt} \|} \not\equiv 0, \quad \liminf_{t\to\infty} h_2(t) \geqslant (1 - \limsup_{t\to\infty} \gamma(t))\,\varepsilon d > 0.$$

This means that Condition 8° holds. We have thus shown that the conditions of Corollary 4 are satisfied, which completes the proof.

Corollary 8. Assume that the following conditions are satisfied:

1) the real parts of all eigenvalues of the matrices B and C are negative;

2) $0 \in \operatorname{int} W(t)$, $t \geqslant 0$, where $W(t) = e^{Bt}U \overset{*}{-} e^{Ct}V$.

Then pursuit from any initial positions may be terminated in a finite time.

Theorem 4. Assume that Condition 9° is satisfied and also:

1) all eigenvalues of the matrices B and C have negative real parts;

2) U is a convex compactum and the zero in the relative interior of U (i.e., the interior relative to the support of U /9/);

3) the system $x^\cdot = Bx + u$, $u \in U$, is completely controllable /10, 11/;

4) the continuous function $\gamma(\cdot)$ is not identically 1.

Then pursuit from any initial positions x° and y° may be terminated in a finite time.

Proof. By Condition 4 of the theorem, there exist numbers $\delta > 0$, $t_2 > t_1 \geqslant 0$ such that $1-\gamma(t) \geqslant \delta$ for all $t \in [t_1, t_2]$. We will prove the existence of a number $\varepsilon > 0$ such that

$$\varepsilon D \subset \int_{t_1}^{t_2} e^{B\tau} U \, d\tau \tag{4.1}$$

Assume the contrary. Then by Condition 2 there exists a vector $\psi \neq 0$ such that

$$\left(\int_{t_1}^{t_2} e^{B\tau} U \, dt, \psi\right) = 0$$

This condition is equivalent to the identity $(e^{B\tau}u, \psi) = 0$ for all $\tau \in [t_1, t_2]$ and $u \in U$. Differentiating this identity k times at the point $\tau \in (t_1, t_2)$, multiplying by $(t-\tau)^k/k!$ $(t \in R)$, and summing over k, we obtain

$$\sum_{k=0}^{\infty} (B^k e^{B\tau}u, \psi) \frac{(t-\tau)^k}{k!} = (e^{Bt}u, \psi) = 0$$

for all $t \in R$ and $u \in U$. This identity contradicts Condition 3. Inclusion (4.1) thus holds.

From Condition 1, for any $x^\circ, y^\circ \in R^n$ there exists a time $T \geqslant t_2$ such that $\| e^{BT}x^\circ - e^{CT}y^\circ \| \leqslant \varepsilon\delta$. Comparing this with (4.1), we conclude that there exists a measurable function $\eta(\tau) \in U$ $(\tau \in [0, T])$ such that

$$\delta \int_{t_1}^{t_2} e^{B\tau}\eta(\tau)\, d\tau = -(e^{BT}x^\circ - e^{CT}y^\circ)$$

For an arbitrary measurable function $v(t) \in V$, we choose the measurable control $u(t) = u(x^\circ, y^\circ, v(t)) \in U$ from the equation

$$e^{B(T-t)}u(t) = \begin{cases} e^{C(T-t)}v(t) + \delta e^{B(T-t)}\eta(T-t), & t \in [T-t_2, T-t_1] \\ e^{C(T-t)}v(t), & t \in [0, T-t_2) \cup (T-t_1, T] \end{cases}$$

which is solvable by the inclusion

$$e^{Bt}U \overset{*}{-} e^{Ct}V \supset (1-\gamma(t))\, e^{Bt}U$$

This control solves the problem of pursuit from the initial positions x° and y° in time T:

$$x(T) - y(T) = e^{BT}x^\circ - e^{CT}y^\circ + \int_0^T e^{B(T-t)}u(t) - e^{C(T-t)}v(t)\,dt =$$

$$e^{BT}x^\circ - e^{CT}y^\circ + \delta \int_{T-t_2}^{T-t_1} e^{B(T-t)}\eta(T-t)\,dt = 0,$$

which it was required to prove.

Remark 2. If we abandon the goal of exact hitting of the terminal set and consider the problem of the arrival of the trajectory of the CCP (1.1) in some (arbitrarily small) ε-neighbourhood of the terminal set, then various conditions on the parameters of the CCP (1.1) may be simplified considerably. By the assertions of /12/, it is sufficient to require the corresponding conditions only for $\varphi(U, v)$ (i.e., for the convex hull). If moreover the CCP (1.1) satifies the "small-game saddle point condition" /13/, the control $u(\cdot)$ can be constructed in the class of piecewise-constant positional controls /14/.

Example 1. Consider a linear pursuit problem which does not satisfy the conditions of Corollary 7 and yet Theorem 2 applies:

$$x' = Bx + Fu,\ y' = v;\ x, y \in R^2;\ u \in U,\ v \in V \tag{4.2}$$

$$U = V = \{u \in R^2 : \|u\| \leqslant 1\},\ B = \begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix},\ F = \begin{Vmatrix} 2, & 0 \\ 0, & 1 \end{Vmatrix}$$

The operator e^{Bt} defines rotation of R^2 with the "velocity" $(2\pi)^{-1}$ and FU is an ellipse with axes of length 1 and 2. Therefore $W(t) = e^{Bt}FU \overset{*}{-} V = 0$ for all $t \geqslant 0$.

If we change to polar coordinates (r, ψ) in R^2 and recall the equation of an ellipse in polar coordinates, we obtain the function (2.2):

$$\beta(t, \psi) = \frac{2}{\sqrt{1 + 3\sin^2(t+\psi)}} 1 - \geqslant 0,\ 0 \leqslant \psi < 2\pi$$

This function is 2π-periodic in t and is almost everywhere positive. Therefore Condition 7° is satisfied for this function. Now applying Theorem 2, we obtain complete controllability of the process (4.2).

Example 2. ("The boy and the crocodile" /8/).

$$z_1' = z_2 - v,\quad z_2' = u;\quad z_1, z_2, u, v \in R^n,\ \|u\| \leqslant 1,\ \|v\| \leqslant 1 \tag{4.3}$$

$$M^* = \{(z_1, z_2) \in R^n \times R^n : \|z_1\| \leqslant 1/2\}$$

Let

$$\omega(t, \tau) = \begin{cases} 0, & \tau \in [0, t-1] \\ 2(\tau + 1 - t), & \tau \in (t-1, t]\ t \geqslant 1 \end{cases}$$

We can verify that the set (1.2) is

$$\Phi(t, \tau) = ((t-\tau)U - \omega(t, \tau)M) \overset{*}{-} V = (t - \tau + \omega(t, \tau)/2 - 1)D$$

Therefore, $0 \in \Phi(t, \tau)$ and the function (1.14) is $g_2(t, \tau) = t - \tau - 1 + 1/_2\omega(t, \tau)$. Therefore the function

$$\int_0^t g_2(t, \tau)\,d\tau = \frac{(t-1)^2}{2}$$

increases without limit as t goes to infinity.

Note that this example does not satisfy Condition 1°, but the "crocodile" (u) may stop for the time $\|z_2^\circ\|$ and then start pursuing the "boy" (v). We may thus take $z_2^\circ = 0$. Combining Corollaries 1 and 3, we obtain controllability of process (4.3).

Example 3. (Pontryagin's test example /8/).

$$z_1' = z_2,\quad z_2' = az_2 + \rho u,\quad z_3' = z_4,\ z_4' = bz_4 + \sigma v \tag{4.4}$$

$$z_1, z_2, z_3, z_4, u, v \in R^n;\quad a, b, \sigma, \rho \in R;\ \|u\| \leqslant 1,\ \|v\| \leqslant 1$$

$$M^* = M^\circ = \{(z_1, z_2, z_3, z_4) : z_1 = z_3\}$$

Performing an analysis similar to that in /8/, we obtain that the function (2.2) in this example has the form

$$\beta(t, z) = \frac{e^{at} - 1}{a} \rho - \frac{e^{bt} - 1}{b} \sigma, \quad t \geqslant 0$$

Then, if $a < 0, b < 0, \rho \geqslant \sigma$, and $\rho/a < \sigma/b$, Theorem 2 is applicable, and it guarantees complete controllability of process (4.4).

REFERENCES

1. GRIGORENKO N.L., The linear problem of the pursuit of several objects, Dokl. Akad. Nauk SSSR, 258, 2, 1981.
2. CHIKRII A.A., Group pursuit with bounded coordinates of the evader, PMM, 46, 6, 1982.
3. CHIKRII A.A., Differential games with several pursuers, Banach Centre Publ. Mathem. Control Theory, PWN, Warsaw, 14, 1985.
4. KERIMOV A.K., On the pursuit problem for one class of linear differential games, Izv. Akad. Nauk SSSR, Tekh. Kibern., 2, 1976.
5. YOUNG J., On differential pursuit games, SIAM J. Contr. Optim. 26, 2, 1988.
6. GANTMAKHER F.R., Matrix Theory, Nauka, Moscow, 1988.
7. WARGA J., Optimal Control of Differential and Functional Equations, Academic Press, New York, 1972.
8. PONTRYAGIN L.S., Linear differential games of pursuit, Matem. Sb., 112, 3, 1980.
9. ROCKAFELLAR R.T., Convex Analysis, Princeton Univ. Press, 1970.
10. BLAGODATSKIKH V.I., Linear Theory of Optimal Control, Izd. MGU, Moscow, 1978.
11. LEE E.B. and MARKUS L., Foundations of Optimal Control Theory, Wiley, New York, 1967.
12. KASKOSZ B., On a non-linear evasion problem, SIAM J. Contr. Optim., 15, 4, 1977.
13. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games, Nauka, Moscow, 1974.
14. CHIKRII A.A. and BELOUSOV A.A., Positional pursuit of a terminal set of a complex structure, in: Non-smooth Problems of Optimization and Control, IMM UrO Akad. Nauk SSSR, Sverdlovsk, 1988.

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THE EXISTENCE AND STABILITY OF INVARIANT SETS OF DYNAMICAL SYSTEMS*

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The possibility of using Lyapunov functions to construct invariant sets of dynamical systems is discussed. The investigations presented herein are based on certain ideas known from the literature /1-11/ and culminate in a generalization of Routh's Theorem and its modification /1-6, 12, 13/.

1. Consider a dynamical system whose behaviour is described by ordinary differential equations of the following form:

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}) \ (\mathbf{x} \in \mathbf{R}^n, \ \mathbf{f}(\mathbf{x}) \in C^1\colon \mathbf{R}^n \to \mathbf{R}^n) \tag{1.1}$$

Assume that Eqs.(1.1) have first integrals which do not depend explicitly on time:

$$\mathbf{U}(\mathbf{x}) = \mathbf{c} \ (\mathbf{c} \in \mathbf{R}^k, \ U(\mathbf{x}) \in C^2\colon \mathbf{R}^n \to \mathbf{R}^k) \tag{1.2}$$

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